Background on D-modules

I. D_X

Let X be a smooth, quasi-proj variety $/\mathbb{C}$

Def The sheaf D_X of $\mathbb{C}$-linear differential operators on X is the $\mathbb{C}$-subalg of $End_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{T}_X = Der_{\mathbb{C}}(\mathcal{O}_X)$

By the smoothness, every point $x \in X$

has an open affine nbhd $U \subset X$ with local coordinates $x_1, \dots, x_n \in \mathcal{O}_X(U)$
$\partial_1, \dots, \partial_n \in \mathcal{T}_X(U)$

such that
$$[\partial_i, \partial_j] = 0 \quad \forall i,j$$
$$[x_i, x_j] = 0 \quad \forall i,j$$
$$[\partial_i, x_j] = \delta_{ij}$$
and
$$\mathcal{T}_X|_U = \bigoplus_{i=1}^{n} \mathcal{O}_U \partial_i$$

Over such a U,
$$\mathcal{D}_X|_U = \bigoplus_{\alpha_1, \dots, \alpha_n \geq 0} \mathcal{O}_U \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

II. $\mathcal{D}_X$-modules left v.s. right

The basic example of a left $\mathcal{D}_X$-module is $\mathcal{O}_X$

" " " " " right " " " is ω_X
$= \Lambda^n \Omega^1_{X/\mathbb{C}}$

$$\omega_X \cong \cancel{\mathcal{H}om}\ \mathcal{H}om_{\mathcal{O}_X}(\Lambda^n T_X, \mathcal{O}_X)$$

If $\omega \in \omega_X$, then $\omega \cdot \delta = -\mathrm{Lie}_\delta(\omega)$

where
$$\mathrm{Lie}_\delta(\omega)(\delta_1 \wedge \cdots \wedge \delta_n) = \delta(\omega(\delta_1 \wedge \cdots \wedge \delta_n)) - \sum_{i=1}^{n} \omega(\delta_1 \wedge \cdots \wedge [\delta, \delta_i] \wedge \cdots \wedge \delta_n)$$

Switching sides:

~~There's an equivalence between cats of left $\mathcal{D}$-modules & right $\mathcal{D}$-modules.~~

$$M \longmapsto \omega_X \otimes_{\mathcal{O}_X} M$$

$$\begin{tikzcd} \text{left } \mathcal{D}_X\text{-module} \arrow[r, bend left] & \text{right } \mathcal{D}_X\text{-module} \arrow[l, bend left] \end{tikzcd}$$

$$\mathcal{H}om_{\mathcal{O}_X}(\omega_X, N)\ \cancel{\mathcal{H}om} \cong \omega_X^{-1} \otimes_{\mathcal{O}_X} N \longleftarrow\!\shortmid N$$

III. Pushing forward & Pulling back

Suppose $f: X \to Y$ is a morphism of smooth quasi-proj varieties $/\mathbb{C}$

Pullback: Let M be a left $\mathcal{D}_Y$-module.

$$f^*M \underset{\substack{\mathcal{O}_X\text{-module}\\ \text{pullback}}}{=\!=\!=} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$$

We can make f^*M a left $\mathcal{D}_X$-module via following:

$$\forall \delta \in T_X, \quad \delta(\underset{\mathcal{O}_X}{p} \otimes \underset{M}{u}) = \delta(p) \otimes u$$

$$+ \, p \cdot df(\delta)(1 \otimes u)$$

"differential of f" : $df : T_X \to f^*T_Y$

(dual of $f^*\Omega^1_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}}$)

If we apply this to $\mathcal{D}_Y$, we get

a left $\mathcal{D}_X$-module : $f^*\mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$
(& right $\mathcal{D}_Y$-module)

we write $\mathcal{D}_{X\to Y}$ for $f^*\mathcal{D}_Y$ "the transfer bimodule"

Rmk: $\forall$ left $\mathcal{D}_Y$-module M,

$$f^*M \underset{\text{left } \mathcal{D}_X\text{-module}}{\cong} \underbrace{\boxed{\mathcal{D}_{X\to Y}}}_{\substack{\uparrow \\ \text{all chain rule} \\ \text{happens here.}}} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}M$$

Pushforwards: easier for right modules.

Suppose M is a right $\mathcal{D}_X$-module.

Then ~~form~~ $M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ is a right $f^{-1}\mathcal{D}_Y$-module

and so $f_*(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$ is a right $\underbrace{f_*f^{-1}\mathcal{D}_Y}_{\uparrow\ \mathcal{D}_Y}$-mod

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we'll actually work with the derived version

pullback $\mathbb{L}f^*$: $D^b(\mathcal{D}_Y^{\text{left}}) \to D^b(\mathcal{D}_X)$

pushforward f_+ $D^b(\mathcal{D}_X^{op}) \to D^b(\mathcal{D}_Y^{op})$

where: $\mathbb{L}f^*(-) : \mathcal{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}(-)$

$$f_+(-) = \mathbb{R}f_*\left((-) \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{D}_{X\to Y}\right)$$

For any $f: X \to Y$, we can factor f as

$$X \xrightarrow[\Gamma_f]{} X \times Y \xrightarrow[pr]{} Y$$

(closed immersion) (smooth)

	$i: X \hookrightarrow Y$ closed immersion	$p: X \to Y$ smooth
Pushforward	both i_* and $\otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ so $i_+ M$ is q.i.s to a single module	you may get a complex (calculate via relative de Rham complex)
Pullback	same bad news use Koszul complex	$\mathbb{L}p^* = p^*$ because $\mathcal{O}_X$ is $p^{-1}\mathcal{O}_Y$ - flat

An example: $\mathbb{A}^n \xhookrightarrow{i} \mathbb{A}^{n+1} \xrightarrow{p} \mathbb{A}^n$

$x_i \qquad x_i, y \qquad x_i$

Let M be a left $\mathcal{D}_{\mathbb{A}^n}$ - module ~~and N~~
and N a left $\mathcal{D}_{\mathbb{A}^{n+1}}$ - module

$i_+ M = \boxed{\text{Next Page}}$

$p_+ N = (0 \to N \xrightarrow{\partial_y} N \to 0)$

$\mathbb{L}p^* M = p^* M$

$\mathbb{L}i^* N = (0 \to \underset{-1}{N} \xrightarrow{y} \underset{0}{N} \to 0)$

$$i_+M \cong \bigoplus_{j\geq 0} \partial_y^j M$$

∂_y acts "obviously"

x_i, ∂_{x_i} commutes with ∂_y^n

y kills M

so y acts on $\partial_y^j u$

as $y \cdot \partial_y^j u = [y, \partial_y^j] u + \partial_y^j \cdot y u$

$= [y, \partial_y^j] u$

IV. Filtration & gradings

$\mathcal{D}_X$ has a filtration $\{F_l \mathcal{D}_X\}$:

In coordinates on U,

$$F_l \mathcal{D}_X|_U = \bigoplus_{|\alpha|\leq l} \mathcal{O}_U \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

$gr\, \mathcal{D}_X = \bigoplus F_l\mathcal{D}_X / F_{l-1}\mathcal{D}_X$ is a

sheaf of commutative $\mathcal{O}_X$-algs.

$[F_l\mathcal{D}_X, F_m\mathcal{D}_X] \subseteq F_{l+m-1}\mathcal{D}_X$

locally, $gr\mathcal{D}_X|_U \simeq \mathcal{O}_U[\xi_1, \dots, \xi_n]$

classes of $\partial_1, \dots, \partial_n$

Globally, $gr\,\mathcal{D}_X \cong \mathrm{Sym}\,\mathcal{T}_X$

One way to think of the cotangent bundle

is $\underline{\mathrm{spec}}\; gr\mathcal{D}_X \xrightarrow{\pi} X$

$\|$

T^*X

A filtered (left) $\mathcal{D}_X$-module M is a qcoh (as $\mathcal{O}_X$-mod) M with ~~below~~ bounded-below increasing exhaustive

filtration $\underbrace{\{F_pM\}}_{\text{qch } \mathcal{O}_X\text{-submod of } M}$ s.t. $(F_l\mathcal{D}_X)\cdot(F_pM) \subseteq F_{l+p}M$

In this case, $gr^F M = \bigoplus F_pM/F_{p-1}M$

is a qcoh $gr\,\mathcal{D}_X$-module

M is ~~qcoh~~ coherent as a left $\mathcal{D}_X$-module
(i.e. qcoh over $\mathcal{O}_X$ &
locally f.g. over $\mathcal{D}_X$)

iff $\exists$ a fillration F on M

s.t. $gr^F M$ is a coherent over $gr\,\mathcal{D}_X$

Such filtration is called $\boxed{\text{a good filtration}}$

If M is coherent as a left $\mathcal{D}_X$-module

we can consider the support of $\left(gr^F M \,\middle|\, \begin{smallmatrix} F \text{ good} \\ \text{on } M \end{smallmatrix}\right)$ pullback to $T^*X \xrightarrow[\pi]{} X$).

This support, which we call the characteristic variety $Ch(M)$ depends only on M, not on the filtration

Ex: If $M = \mathcal{D}_X$, then $ch(M) = T^*X$

If $M = \mathcal{O}_X$ (or any vector bundle)

then $ch(M) = 0$-section of T^*X

Thm (Bernstein's inequality)

$\forall$ coherent nonzero M

$$\dim ch(M) \geq \dim(X)$$

"=" then M is called $\boxed{\text{holonomic.}}$